MATH 1A - MOCK FINAL - SOLUTIONS

PEYAM RYAN TABRIZIAN

1. *(20 points)* Use the **definition** of the integral to evaluate:

$$\int_1^2 x^2 dx$$

You may use the following formulas:

$$\sum_{i=1}^n 1 = n \qquad \sum_{i=1}^n i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^n i = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$$

Preliminary work:

- $f(x) = x^2$
- $a = 1$, $b = 2$, $\Delta x = \frac{2-1}{n} = \frac{1}{n}$
- $x_i = 1 + \frac{i}{n}$

Date: Friday, August 5th, 2011.

$$
\begin{aligned}
\int_1^2 x^2 dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{n}\right)\left(1+\frac{i}{n}\right)^2 \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{n}\right)\left(1+\frac{2i}{n}+\frac{i^2}{n^2}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}+\frac{2i}{n^2}+\frac{i^2}{n^3} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}+\sum_{i=1}^{n}\frac{2i}{n^2}+\sum_{i=1}^{n}\frac{i^2}{n^3} \\
&= \lim_{n\to\infty} \frac{1}{n}\left(\sum_{i=1}^{n} 1\right)+\frac{2}{n^2}\left(\sum_{i=1}^{n} i\right)+\frac{1}{n^3}\left(\sum_{i=1}^{n} i^2\right) \\
&= \lim_{n\to\infty} \frac{1}{n}(n)+\frac{2}{n^2}\left(\frac{n(n+1)}{2}\right)+\frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= \lim_{n\to\infty} 1+\frac{n+1}{n}+\frac{(n+1)(2n+1)}{6n^2} \\
&=1+1+\frac{2}{6} \\
&=2+\frac{1}{3} \\
&=\frac{7}{3}
\end{aligned}
$$

Check: (not required, but useful)

$$
\int_1^2 x^2 dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{8}{3}-\frac{1}{3}=\frac{7}{3}
$$

2. *(10 points)* Evaluate the following limit:

$$\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}\right)$$

Preliminary work:

- $f(x) = \sqrt{x}$
- $x_i = \frac{i}{n}$
- $a = x_0 = 0, b = x_n = 1$

Hence the limit equals to:

$$\int_0^1 \sqrt{x}dx = \int_0^1 x^{\frac{1}{2}}dx = \left[\frac{x^3 2}{\frac{3}{2}}\right]_0^1 = \frac{1}{\frac{3}{2}} - 0 = \frac{2}{3}$$

3. *(50 points, 5 points each)* Find the following:

(a) The antiderivative F of $f(x) = x^2 + 3x^3 - 4x^7$ which satisfies $F(0) = 1$

The MGAD of f is:

$$F(x) = \frac{x^3}{3} + \frac{3x^4}{4} - \frac{4x^8}{8} + C = \frac{x^3}{3} + \frac{3}{4}x^4 - \frac{1}{2}x^8 + C$$

To solve for C, use the fact that $F(0) = 1$, so $0+0-0+C = 1$, so $\boxed{C = 1}$, and hence:

$$F(x) = \frac{x^3}{3} + \frac{3}{4}x^4 - \frac{1}{2}x^8 + 1$$

(b) $\int_{-1}^{1} |x| \, dx$

If you draw a picture of $f(x) = |x|$, you should notice that the integral is the sum of two triangles, one with base 1 and height 1 (from -1 to 0) and the other one with base 1 and height 1 (from 0 to 1), hence we get:

$$\int_{-1}^{1} |x| \, dx = \frac{1}{2}(1)(1) + \frac{1}{2}(1)(1) = \frac{1}{2} + \frac{1}{2} = 1$$

(c)

$$\int_{-\pi}^{\pi} \sin(x)(1 + \cos(x) + e^{x^2} + 42x^{2012})dx = 0$$

Since the function is an odd function!

(d)

$$\int x^2 + 1 + \frac{1}{x^2+1} dx = \frac{x^3}{3} + x + \tan^{-1}(x) + C$$

(e) $\int_1^e \frac{(\ln(x))^2}{x} dx$

Let $u = \ln(x)$, then $du = \frac{1}{x}dx$, and $u(1) = \ln(1) = 0$, and $u(e) = \ln(e) = 1$, so:

$$\int_1^e \frac{(\ln(x))^2}{x} dx = \int_0^1 u^2 du = \left[\frac{u^3}{3}\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

(f)

$$\begin{aligned}\int_\pi^{2\pi} (\cos(x) - 2\sin(x))\, dx =& [\sin(x) + 2\cos(x)]_\pi^{2\pi} \\ =& \sin(2\pi) + 2\cos(2\pi) - \sin(\pi) - 2\cos(\pi) \\ =& 0 + 2 - 0 + 2 \\ =& 4\end{aligned}$$

(g) $g'(x)$, where $g(x) = \int_x^{e^x} \sqrt{1+t^2} dt$

Let $f(t) = \sqrt{1+t^2}$, then $g(x) = F(e^x) - F(x)$, so:

$$g'(x) = F'(e^x)e^x - F'(x) = f(e^x)e^x - f(x) = \sqrt{1+(e^x)^2}(e^x) - \sqrt{1+x^2}$$

(h)

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \frac{1+\cos^2(\theta)}{\cos^2(\theta)} d\theta =& \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2(\theta)} + 1 d\theta \\ =& \int_0^{\frac{\pi}{4}} \sec^2(\theta) + 1 d\theta \\ =& [\tan(\theta) + \theta]_0^{\frac{\pi}{4}} \\ =& \tan(\frac{\pi}{4}) + \frac{\pi}{4} - \tan(0) - 0 \\ =& 1 + \frac{\pi}{4}\end{aligned}$$

(i) $\int e^x \sqrt{1+e^x} dx$

Let $u = 1 + e^x$, then $du = e^x dx$, so:

$$\int e^x \sqrt{1+e^x} dx = \int \sqrt{u} du = \frac{2}{3} u^{\frac{3}{2}} + C = \frac{2}{3}(1+e^x)^{\frac{3}{2}} + C$$

(Don't forget to substitute $u = 1 + e^x$ back into your integral!)

(j) The average value of $f(x) = \sin(x)$ on $[-\pi, \pi]$

$$\frac{\int_{-\pi}^{\pi} \sin(x) dx}{\pi - (-\pi)} = \frac{0}{2\pi} = 0$$

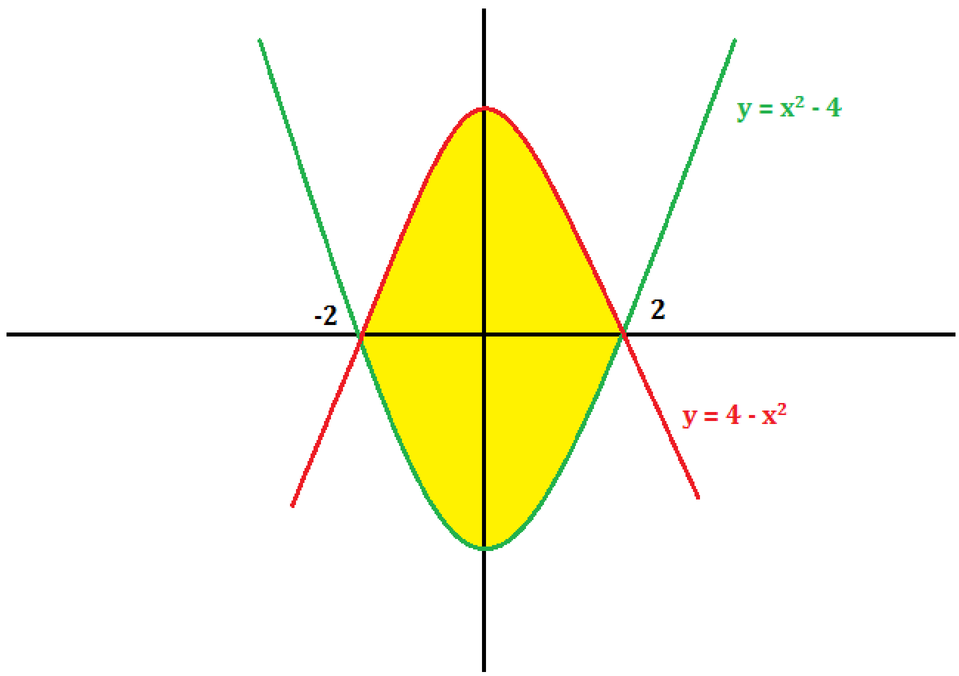
Since $\sin(x)$ is an odd function!

4. *(20 points)* Find the area of the region enclosed by the curves:

$$y = x^2 - 4 \quad \text{and} \quad y = 4 - x^2$$

First draw a picture:

1A/Math 1A Summer/Exams/MockFparabola.png



Then determine the points of intersection between the two parabolas:

$$\begin{aligned} x^2 - 4 =& 4 - x^2 \\ 2x^2 =& 8 \\ x^2 =& 4 \\ x =& \pm 2 \end{aligned}$$

And notice that on $[-2, 2]$, $4 - x^2$ is always above $x^2 - 4$, so the area of the region is:

$$\begin{aligned}\int_{-2}^{2}(4-x^2)-(x^2-4)dx &= \int_{-2}^{2} 8-2x^2dx\\ &= \left[8x-\frac{2}{3}x^3\right]_{-2}^{2}\\ &= 16-\frac{2}{3}(8)-(-16+\frac{2}{3}(8))\\ &= 16-\frac{16}{3}+16-\frac{16}{3}\\ &= 32-\frac{32}{3}\\ &= \frac{64}{3}\end{aligned}$$

Of course, if you're clever about this, you might have noticed that the area is $4\int_0^2 4-x^2dx$, but you didn't have to be so clever about it! :)

5. *(20 points, 5 points each)* Find the following limits

(a)

$$
\begin{aligned}
\lim_{x\to\infty} \sqrt{x^2+x} - x &= \lim_{x\to\infty} \frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x} \\
&= \lim_{x\to\infty} \frac{x^2+x-x^2}{\sqrt{x^2+x}+x} \\
&= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x} \\
&= \lim_{x\to\infty} \frac{x}{\sqrt{x^2}\sqrt{1+\frac{1}{x}}+x} \\
&= \lim_{x\to\infty} \frac{x}{x\sqrt{1+\frac{1}{x}}+x} \qquad \text{since } \sqrt{x^2} = |x| = x, \text{ since } x > 0 \\
&= \lim_{x\to\infty} \frac{x}{x\left(\sqrt{1+\frac{1}{x}}+1\right)} \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}+1} \\
&= \frac{1}{1+1} \\
&= \frac{1}{2}
\end{aligned}
$$

(b) $\lim_{x\to\infty}(1+x)^{\frac{1}{x}}$

1) Let $y = (1+x)^{\frac{1}{x}}$

2) Then $\ln(y) = \frac{1}{x}\ln(1+x) = \frac{\ln(1+x)}{x}$

3)

$$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} \frac{\ln(1+x)}{x} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+x}}{1} = 0$$

4) Hence

$$\lim_{x\to\infty} y = e^0 = 1$$

(c) $\lim_{x\to 0} xe^{\sin\left(\frac{1}{x}\right)}$

First of all,

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

Hence:

$$e^{-1} \leq e^{\sin\left(\frac{1}{x}\right)} \leq e^1$$

And so:

$$xe^{-1} \leq xe^{\sin\left(\frac{1}{x}\right)} \leq xe$$

But $\lim_{x\to 0} xe^{-1} = \lim_{x\to 0} xe = 0$, hence **by the Squeeze theorem**,

$$\lim_{x\to 0} xe^{\sin\left(\frac{1}{x}\right)} = 0$$

(d)

$$\lim_{x\to\infty} \frac{(\ln(x))^2}{x} \stackrel{H}{=} \lim_{x\to\infty} \frac{2\ln(x)\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2\ln(x)}{x} \stackrel{H}{=} \lim_{x\to\infty} \frac{\frac{2}{x}}{1} = \frac{2}{\infty} = 0$$

6. *(20 points, 5 points each)* Find the derivatives of the following functions

(a) $f(x) = \sin(x)e^{\tan(x)}$

$$f'(x) = \cos(x)e^{\tan(x)} + \sin(x)e^{\tan(x)}\sec^2(x)$$

(b) $f(x) = x^{\cos(x)}$

Logarithmic differentiation

1) Let $y = x^{\cos(x)}$

2) Then $\ln(y) = \cos(x)\ln(x)$

3) $\frac{y'}{y} = -\sin(x)\ln(x) + \frac{\cos(x)}{x}$

4)

$$y' = y\left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right) = x^{\cos(x)}\left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right)$$

(c) y', where $x^3 + y^3 = xy$

$$\begin{aligned} 3x^2 + 3y^2y' &= y + xy' \\ 3y^2y' - xy' &= y - 3x^2 \\ (3y^2 - x)y' &= y - 3x^2 \\ y' &= \frac{y - 3x^2}{3y^2 - x} \end{aligned}$$

(d) y' at $(0, 1)$, where $\frac{x^2+y^2}{x^2-y^2} = -y$

$$\frac{(2x + 2yy')(x^2 - y^2) - (x^2 + y^2)(2x - 2yy')}{(x^2 - y^2)}^2 = -y'$$

Now plug in $x = 0$ and $y = 1$

$$\begin{aligned}\frac{2y'(-1) - (1)(-2y')}{1} &= -y' \\ -2y' + 2y' &= -y' \\ 0 &= -y' \\ y' &= 0\end{aligned}$$

7. *(10 points)* Find the absolute maximum and minimum of the following function on $[0, \pi]$:

$$f(x) = x + \cos(x)$$

1) Endpoints: $f(0) = 1$, $f(\pi) = \pi - 1$

2) Critical numbers:

$$f'(x) = 1 - \sin(x) = 0 \Longleftrightarrow \sin(x) = 1 \Longleftrightarrow x = \frac{\pi}{2}$$

$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \cos\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

3) Compare: The absolute max of f is $\boxed{f(\pi) = \pi - 1}$ and the absolute min of f is $\boxed{f(0) = 1}$

Bonus 1 *(5 points)* Show that if f is continuous on $[0,1]$, then $\int_0^1 f(x)dx$ is bounded, that is, there are numbers m and M such that:

$$m \leq \int_0^1 f(x)dx \leq M$$

Hint: Use one of the 'value' theorems that we haven't used a lot in this course (see section 4.1)

By the extreme value theorem, f attains an absolute max M and an absolute min m. This means that for all x in $[0,1]$:

$$m \leq f(x) \leq M$$

Now integrate:

$$\int_0^1 m dx \leq \int_0^1 f(x)dx \leq \int_0^1 M dx$$

$$m(1-0) \leq \int_0^1 f(x)dx \leq M(1-0)$$

$$m \leq \int_0^1 f(x)dx \leq M$$

Bonus 2 *(5 points)* If $f(x) = Ax^3 + Bx^2 + Cx + D$ is a polynomial such that:

$$\frac{A}{4} + \frac{B}{3} + \frac{C}{2} + D = 0$$

Show that f has at least one zero on $(0, 1)$.

Hint: What is the *average* value of f on $[0, 1]$?

By the MVT for integrals on $[0, 1]$, for some c in (0,1), we have:

$$f(c) = \frac{\int_0^1 f(x)dx}{1 - 0}$$

But:

$$\begin{aligned}
\frac{\int_0^1 f(x)dx}{1-0} &= \int_0^1 f(x)dx \\
&= \int_0^1 (Ax^3 + Bx^2 + Cx + D)dx \\
&= \left[\frac{A}{4}x^4 + \frac{B}{3}x^3 + \frac{C}{2}x^2 + Dx\right]_0^1 \\
&= \frac{A}{4} + \frac{B}{3} + \frac{C}{2} + D \\
&= 0
\end{aligned}$$

Hence, for some c in (0,1), we have $\boxed{f(c) = 0}$, so f has at least one zero c in $(0, 1)$.

Bonus 3 *(5 points)* Another way to define $\ln(x)$ is:

$$\ln(x) = \int_1^x \frac{1}{t} dt$$

Show **using this definition only** that for all a and b:

$$\ln(ab) = \ln(a) + \ln(b)$$

Hint: Fix a constant a, and consider the function:

$$g(x) = \ln(ax) - \ln(x) - \ln(a)$$

$$\begin{aligned} g(x) =& \ln(ax) - \ln(x) - \ln(a) \\ =& \int_1^{ax} \frac{1}{t} dt - \int_1^{x} \frac{1}{t} dt - \int_1^{a} \frac{1}{t} dt \\ =& F(ax) - F(1) - (F(x) - F(1)) - (F(a) - F(1)) \end{aligned}$$

Where F is an antiderivative of $f(t) = \frac{1}{t}$

Now differentiating g, and using the fact that a is a constant, we get:

$$\begin{aligned} g'(x) =& F'(ax)(a) - 0 - F'(x) + 0 - 0 + 0 \\ =& f(ax)(a) - f(x) \\ =& \left(\frac{1}{ax}\right)(a) - \frac{1}{x} \\ =& \frac{1}{x} - \frac{1}{x} \\ =& 0 \end{aligned}$$

Hence $g'(x) = 0$, so $g(x) = C$, where C is a constant.

To figure out what C is, let's calculate $g(1)$:

$$\begin{aligned} g(1) &= C \\ \int_1^1 \frac{1}{t} dt &= C \\ 0 &= C \\ C &= 0 \end{aligned}$$

Hence $C = 0$, and so $g(x) = 0$, whence $\ln(ax) - \ln(x) - \ln(a) = 0$, so $\ln(ax) = \ln(a) + \ln(x)$.

Since this holds for all x, let $x = b$, and we get:

$$\ln(ab) = \ln(a) + \ln(b)$$

BAZINGA!!!